Čech Cohomology in Homotopy Type Theory

Ingo Blechschmidt, Felix Cherubini and David Wärn

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Contents

Introduction

This is an incomplete draft on work in progress.

In pure mathematics, it is a common practice to simplify questions about complicated objects by assigning them more simple objects in a systematic way that faithfully represents some features of interest. One particular, but still surprisingly broad applicable instantiation of this appraoch, is the assignment of a sequence of abelian groups, the cohomology groups, to spaces, sheaves and other things. Over the last century, cohomology was first discovered in concrete examples, then generalized and streamlined – a process that culminated in the presentation of cohomology groups as the connected components in mapping spaces in higher toposes.

This is a representation, we can easily and elementary use through the interpretation of homotopy type theory in higher toposes. In [Cav15] results about cohomology theories like the Mayer-Vietoris-Sequence were proven and computations were carried out, in [van18] the Serre-Spectral-Sequence was constructed and used. The latter also introduced cohomology with non-constant coefficients, which are the right level of generality for the applications we have in mind. We are particularly interested in computing cohomology groups of sheaves in algebraic geometry, which can be done synthetically using the foundation laid out by [CCH23] building on work and ideas of Ingo Blechschmidt ([Ble17]), Anders Kock ([Koc06]) and David Jaz Myers ([Mye19b], [Mye19a]).

In this setup, the basic spaces in algebraic geometry, schemes, are just sets with a particular property [CCH23][def of scheme], and instead of sheaves on a type X, we consider, more generally maps $A : X \to \mathrm{Ab}$ to the type of abelian groups. The cohomology groups are then defined as dependent function types with values in Eilenberg-MacLane-Spaces

$$H^n(X, A) :\equiv \|(x : X) \to K(A(x), n)\|_0$$

– a definition first suggested by [Shu13]. Due to its simplicity, this is very convenient to work with. One common way to calculate cohomology groups $H^n(X, \mathcal{F})$ is to use results about the cohomology of simple subspaces $U_i \subseteq X$. A computational result on the case with two subspaces $U, V \subseteq X$ is known as the *Mayer-Vietoris-Sequence*. In general this sequence helps to calculate the cohomology groups of a pushout and was constructed for cohomology with constant coefficients in a group in [Cav15]. We generalize this result to non-constant coefficients (lemma 1.0.12) with a slick proof the second author learned in parts from Urs Schreiber in the course of his PhD-thesis.

Čech Cohomology, in the sense of this work, is a generalization of the Mayer-Vietoris Sequence in the case, where U, V are actually subtypes of a set, to a space X which is the union of fintely many subtypes $U_i \subseteq X$, i.e. $\bigcup_i U_i = X$. From a *synthetic homotopy theory*, this is not very interesting, but it is very interesting for our intended applications in *synthetic algebraic geometry*. In the latter subhject, it was unclear for a long time how one could set up a theory of cohomology, since the classical treatment relies on (non-)constructions, which need the axiom of choice.

In [CCH23] this problem is circumvented, by using a justified axiom which allows a bit of choice which is related to the topology of the relevant topos and, secondly, as mentioned above, by using higher types to define work with cohomology.

We present two approaches to a proof of a sufficiently general isomorphism between Čech Cohomology groups and cohomology groups defined using higher types. The first appraoch is more conceptual, more general and makes use of the higher types with have available in HoTT. It is also related to how one would produce a Čech Cohomology theorem in higher category theory: the space is represented as a colimit, so mapping into the coefficients should yield a limit description of the (untruncated) cohomology of the whole space.

The second approach very roughly follows old classical treatments of Grothendieck ([Gro57]) and Buchsbaum ([Buc60]). Inspired by this, we aim to show that both cohomology defined using higher types and Čech cohomology satisfies the universal property of a *universal ∂-functor* in some furtunate but still quite relevant situations. While this approach is far less general, it also seems to need far less involved calculations.

Acknowledgements

The second author wants to thank Urs Schreiber, who taught him the nice pullback pasting tricks, we use to construct the Mayer–Vietoris sequence. He also wants to thank Tobias Columbus, who made him aware of Buchsbaum's work and with whom he had many discussions about early versions of the approach with ∂-functors, and, more from the distant past, he thanks Michael Fütterer for presenting the idea of ∂-functors so clearly to him.

1 The Mayer-Vietoris-Sequence

TODO: This needs to be updated and made coherent. It might be good to remove connective spectra and the connective cover (not sure if they are still needed).

An n-th delooping of a pointed space A which is also $(n-1)$-connected is unique and usually written as $B^n A$ or $K(A, n)$ and called an *Eilenberg-MacLane space*. We will just write A_n for an n-th delooping.

It is known, that in HoTT, a (0-truncated) abelian group can be delooped arbitrarily often ([LF14]).

Contents of this section are from Mike Shulman's posts on the HoTT-Blog about cohomology, Floris van Doorn's thesis ([van18])[section 5.3] and common knowledge in the field that is not written up, with the possible exception of the Mayer-Vietoris-Sequence with non-constant coefficients (lemma 1.0.12).

Suppose we have a pointed type A with delooping A_k for any $k : \mathbb{N}$. Then, analogous to the definition of the k-th homotopy group

$$\pi_k :\equiv \|\Omega^k A\|_0$$

one could define homotopy groups of *negative* degree $-k$ by:

$$\pi_{-k} :\equiv \|A_k\|_0$$

Note that these will be trivial for any Eilenberg-MacLane spectrum, since for those, A_{k+1} is k-connected for $k : \mathbb{N}$. In general, spectra with trivial homotopy groups in negative degree are called *connective*. The result in this article is concerned with Eilenberg-MacLane spectra.

We will use spectra varying over a space as coefficints for cohomology, which corresponds to the classical concept of parametrized spectra. We fix our terminology in the following definition.

Definition 1.0.1 (a) A *spectrum* is a sequence of pointed types $(A_k)_{k:\mathbb{N}}$, together with pointed equivalences $A_k \simeq \Omega A_{k+1}$.

(b) A spectrum $(A_k)_{k:\mathbb{N}}$ is *connective*, if $\|A_{k+1}\|_0 \simeq 1$ for all $k : \mathbb{N}$.

(c) Let X be a type. A *parametrized spectrum* over X, is a dependent function, which assigns to any $x : X$, a spectrum $(A_{x,k})_{k:\mathbb{N}}$. For brevity, We will call a parametrized spectrum $A \equiv x \mapsto (A_{x,k})_{k:\mathbb{N}}$ over X just *spectrum over* X.

(d) A *morphism of spectra* A, A' over X, is given by a sequence of pointed maps $f_{x,k} : A_{x,k} \to A'_{x,k}$ for any $x : X$, such that $\Omega f_{x,k+1} = f_{x,k}$ (using the pointed equivalences).

The connective spectra form a nice "subcategory": We will need the following (coreflective) construction that turns a spectrum into a connective spectrum. See **??** for the definition of the k-connected cover "$D^k_X d$".

Definition 1.0.2 For a spectrum A, the following construction is called the *connective cover*:

$$\hat{A} :\equiv k \mapsto D^{k-1}_{A,k}$$

There is also a sequence of pointed maps $\varphi_k : \hat{A}_k \to A_k$, given by the projection from the connected covers.

The following fact will be useful to us on various occations and can be proven using the uniqueness of Eilenberg-MacLane spaces:

Lemma 1.0.3 Let X be a type and $A : X \to \mathrm{Ab}$ a dependent abelian group. If for all $0 < l \leq n$ the type $(x : X) \to K(A_x, l)$ is connected, then

$$((x : X) \to K(A_x, n)) = K((x : X) \to A_x, n).$$

Definition 1.0.4 The k-th cohomology group of X with coefficients in A is the following:

$$H^k(X, A) :\equiv \|(x : X) \to A_{x,k}\|_0$$

(Add a disclaimer: pullback and push forward do not coincide with the classical constructions)

Definition 1.0.5 Let $f : Y \to X$ be a map of types and $\mathcal{F} : X \to \mathrm{Ab}$ and $\mathcal{G} : Y \to \mathrm{Ab}$ dependent abelian groups.

(a) $f^*\mathcal{F} :\equiv (y : Y) \mapsto \mathcal{F}_{f(y)}$ is called the *pullback* of $\mathcal{F}$ along f.

(b) $f_*\mathcal{G} :\equiv (x : X) \mapsto (((y, \_) : \mathrm{fib}_f(x)) \to \mathcal{G}_y)$ is called the *push-forward* of $\mathcal{G}$ along f.

Cohomlogy commutes with finite coproducts:

Lemma 1.0.6 Let Y_i, $i : I :\equiv \{1, \ldots, l\}$ be types and $f_i : (i : I) \times Y_i \to X$ and $\mathcal{F} : ((i : I) \times Y_i) \to \mathrm{Ab}$. Then for all $n : \mathbb{N}$

$$H^n((i : I) \times Y_i, \mathcal{F}) = \bigoplus_i H^n(Y_i, f_i^*\mathcal{F}).$$

Proof Direct by currying, using that $\|\_\|_0$ preserves finite products. □

Cohomology does not change under push-forward along maps with cohomologically trivial fibers:

Lemma 1.0.7 Let $f : Y \to X$ and $\mathcal{F} : Y \to \mathrm{Ab}$ be such that $H^l(\mathrm{fib}_f(x), \pi_1^*\mathcal{F}) = 0$ for all $0 < l \leq n$, then

$$H^n(Y, \mathcal{F}) = H^n(X, f_*\mathcal{F}).$$

Proof By direct application of Lemma 1.0.3. □

An important notion in abelian categories, is that of short exact sequences. And it is important to us here, since for every short exact sequence (somewhere), there should be an induced long exact sequence on cohomology groups. The cokernel of an exact sequence, corresponds to a cofiber of a map of spectra.

Definition 1.0.8 Let $f : A \to A'$ be a map of spectra.

(a) The *cofiber* of f is given by the spectrum

$$C_{f,k} :\equiv \mathrm{fib}_{f_{k+1}}$$

together with the map $c : A' \to C_f$, where c_k is induced in the following diagram of pullback-squares:

$$\begin{tikzcd}
A_k \arrow[r, "f_k"] \arrow[d] & A'_k \arrow[r] \arrow[d, dashed] & 1 \arrow[d] \\
1 \arrow[r] & C_{f,k} \arrow[r] \arrow[d] & A_{k+1} \arrow[d] \\
& 1 \arrow[r] & A'_{k+1}
\end{tikzcd}$$

(b) The *fiber* of f is given by the spectrum

$$\mathrm{fib}_{f,k} :\equiv \mathrm{fib}_{f_k}$$

Note that $f : A \to A'$ is always the fiber of its cofiber and conversely, $f : A \to A'$ is always the cofiber of its fiber, which is very different from the situation in a general abelian category, where for example not every map is the kernel of its cokernel.

Definition 1.0.9 A sequence of morphisms of spectra over X

$$A \xrightarrow{f} A' \xrightarrow{g} A''$$

is a *fiber sequence*, if the following equivalent statements hold:
(a) f_x is the fiber of g_x for all $x : X$

(b) $f_{x,k}$ is the fiber of $g_{x,k}$ for all $x : X$ and $k : \mathbb{N}$
If all spectra involved are Eilenberg-MacLane spectra, we call the sequence *exact*, and vice versa, if we speak of a *short exact sequence* of spectra (over X), we assume all spectra involved are Eilenberg-MacLane and we have a fiber sequence.

Lemma 1.0.10 If $A \to A' \to A''$ is a fiber sequence, then the induced square:

$$\begin{tikzcd}
\prod_{x:X} A_{x,k} \arrow[r] \arrow[d] & \prod_{x:X} A'_{x,k} \arrow[d] \\
1 \arrow[r] & \prod_{x:X} A''_{x,k}
\end{tikzcd}$$

is a pullback square for all $k : \mathbb{N}$.

Proof $\prod$ maps families of pullback squares to a pullback square. □

This is just tailored to prove the following proposition:

Proposition 1.0.11 For any fiber sequence

$$A \to A' \to A''$$

of spectra over X, there is a long exact sequence of cohomology groups:

$$\begin{tikzcd}
& \ldots \arrow[r] & H^{n-1}(X, A'') \arrow[dll] \\
H^n(X, A) \arrow[r] & H^n(X, A') \arrow[r] & H^n(X, A'') \arrow[dll] \\
H^{n+1}(X, A) \arrow[r] & H^{n+1}(X, A') \arrow[r] & \ldots
\end{tikzcd}$$

Proof Apply homotopy fiber sequence to last proposition for all $n : \mathbb{N}$. $\square$

Lemma 1.0.12 Let $\mathcal{F}$ be a spectrum on X and assume we have a pushout square of spaces

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "g"] & U \arrow[d, "h"] \\ V \arrow[r, "k"] & X \end{tikzcd}$$

Then we have a Mayer-Vietoris sequence:

$$\begin{tikzcd} & \dots \arrow[r] & H^{n-1}(S, f^*h^*\mathcal{F}) \arrow[dll] \\ H^n(X, \mathcal{F}) \arrow[r] & H^n(U, h^*\mathcal{F}) \oplus H^n(V, k^*\mathcal{F}) \arrow[r] & H^n(S, f^*h^*\mathcal{F}) \arrow[dll] \\ H^{n+1}(X, \mathcal{F}) \arrow[r] & \dots & \end{tikzcd}$$

Proof The square

$$\begin{tikzcd} \prod \mathcal{F} \arrow[r] \arrow[d] & \prod h^*\mathcal{F} \arrow[d] \\ \prod k^*\mathcal{F} \arrow[r] & \prod f^*h^*\mathcal{F} \end{tikzcd}$$

is a pullback by [Rij19, Proposition 2.1.6]. This can be transformed to the following pullback square:

$$\begin{tikzcd} \prod \mathcal{F} \arrow[r] \arrow[d] & \prod f^*h^*\mathcal{F} \arrow[d, "\Delta"] \\ \prod k^*\mathcal{F} \times \prod h^*\mathcal{F} \arrow[r, "\times"] & \prod f^*h^*\mathcal{F} \times \prod f^*h^*\mathcal{F} \end{tikzcd}$$

By [Wel17, Lemma 3.3.6] and the weak group structure on $\prod f^*h^*\Omega^{-n}\mathcal{F}$, we have a pullback square for each $n : \mathbb{N}$:

$$\begin{tikzcd} \prod f^*h^*\Omega^{-n}\mathcal{F} \arrow[r] \arrow[d] & 1 \arrow[d] \\ (\prod f^*h^*\Omega^{-n}\mathcal{F}) \times (\prod f^*h^*\Omega^{-n}\mathcal{F}) \arrow[r, "-"] & \prod f^*h^*\Omega^{-n}\mathcal{F} \end{tikzcd}$$

Pasting gives a fiber-square:

$$\begin{tikzcd} (\prod k^*\Omega^{-n}\mathcal{F}) \times (\prod h^*\Omega^{-n}\mathcal{F}) \arrow[d] & \prod \Omega^{-n}\mathcal{F} \arrow[l] \arrow[d] \\ (\prod f^*h^*\Omega^{-n}\mathcal{F}) \times (\prod f^*h^*\Omega^{-n}\mathcal{F}) \arrow[d, "-"] & \prod f^*h^*\Omega^{-n}\mathcal{F} \arrow[l] \arrow[d] \\ \prod f^*h^*\Omega^{-n}\mathcal{F} & 1 \arrow[l] \end{tikzcd}$$

So we get the desired fiber long exact sequence again by taking the long exact sequence of homotopy groups. $\square$

2 Čech cohomology

In this section, let X be a type, $U_1, \dots, U_n \subseteq X$ open subtypes that cover X and $\mathcal{F} : X \to \mathrm{Ab}$ a dependent abelian group on X. We start by repeating the classical definition of Čhech-Cohomology groups for a given cover.

Definition 2.0.1 (a) For open $U \subseteq X$, we use the notation

$$\mathcal{F}(U) :\equiv \prod_{x:U} \mathcal{F}_x.$$

(b) For $s : \mathcal{F}(U)$ and open $V \subseteq U$ we use the notation $s :\equiv s_{|V} :\equiv (x : V) \mapsto s_x$.

(c) For a selection of indices $i_1, ..., i_l : \{1, \ldots, n\}$, we use the notation

$$U_{i_1 \ldots i_l} :\equiv U_{i_1} \cap \cdots \cap U_{i_l}.$$

(d) For a list of indices $i_1, \ldots, i_l$, let $i_1, \ldots, \hat{i_t}, \ldots, i_l$ be the same list with the t-th element removed.

(e) For $k : \mathbb{Z}$, the k-th *Čech-boundary operator* is the homomorphism

$$\partial^k : \bigoplus_{i_0,\ldots,i_k} \mathcal{F}(U_{i_0 \ldots i_k}) \to \bigoplus_{i_0,\ldots,i_{k+1}} \mathcal{F}(U_{i_0 \ldots i_{k+1}})$$

given by $\partial^k(s) :\equiv (l_0, \ldots, l_{k+1}) \mapsto \sum_{j=0}^{k} (-1)^j s_{l_0,\ldots,\hat{l_j},\ldots,l_k | U_{l_0,\ldots,l_{k+1}}}$.

(f) The k-th *Čech-Cohomology group* for the cover $U_1, \ldots, U_n$ with coefficients in $\mathcal{F}$ is

$$\check{H}^k(\{U\}, \mathcal{F}) :\equiv \ker \partial^k / \operatorname{im}(\partial^{k-1}).$$

Definition 2.0.2 Let $\{U\}_{i:I}$ be a finite collection of open subtypes of X and $\mathcal{F} : X \to \mathrm{Ab}$. Let $I^k_x :\equiv (i_0, \ldots, i_k : I) \times U_{i_0 \ldots i_k}(x)$ for $k : \mathbb{N}$ and $i^k : (x : X) \times I^k_x \to X$ be the first projection. Then the dependent abelian group

$$\check{\mathcal{F}}^k :\equiv (x : X) \mapsto \mathcal{F}_x^{I^k_x} \equiv i^k_* i^{k*} \mathcal{F}$$

is called the k-th *Čech-sheaf* of $\mathcal{F}$.

Remark 2.0.3 (a) The functor $\prod : \mathcal{A} \to \mathrm{Ab}$ is additive.

(b) Let X be a type covered by $\{U\}_{i:I}$ and $\mathcal{F} : X \to \mathrm{Ab}$. Then

$$\prod_{x:X} \check{\mathcal{F}}^k_x = \bigoplus_{i_0,\ldots,i_k} \mathcal{F}(U_{i_0 \ldots i_k}).$$

Proof (a) The finite biproducts in $\mathcal{A}$ are in particular finite products, which commute with $\prod$.

(b)

$$\begin{aligned}
\prod_{x:X} \check{\mathcal{F}}^k_x &= \prod_{x:X} \mathcal{F}_x^{I^k_x} \\
&= \prod_{x:X} ((i_0, \ldots, i_k : I) \times U_{i_0 \ldots i_k}(x)) \to \mathcal{F}_x \\
&= \prod_{x:X} \prod_{i_0,\ldots,i_k:I} \mathcal{F}_x^{U_{i_0 \ldots i_k}(x)} \\
&= \prod_{i_0,\ldots,i_k:I} \prod_{x:X} \mathcal{F}_x^{U_{i_0 \ldots i_k}(x)} \\
&= \bigoplus_{i_0,\ldots,i_k} \mathcal{F}(U_{i_0 \ldots i_k}).
\end{aligned}$$

□

Definition 2.0.4 (a) A cover $\{U\} = U_1, \ldots, U_n$ is called *acyclic* for $\mathcal{F}$ if for all $k : \mathbb{N}$ and $i_0, \ldots, i_k$, we have that the higher (non Čech) cohomology groups are trivial:

$$\forall l > 0. H^l(U_{i_0,\ldots,i_k}, \mathcal{F}) = 0.$$

(b) A cover $\{U\} = U_1, \ldots, U_n$ is called *Čech-trivializing* (Better names welcome!) for $\mathcal{F}$ if for all $l > 0$, $k \geq 0$ and indices $i_o, \ldots, i_n : I$ we have $H^1(U_{i_0 \ldots i_k}, \mathcal{F}) = 0$ and $H^1(U_{i_0 \ldots i_k}(x), \mathcal{F}_x) = 0$ for all $x : X$.

Theorem 2.0.5
Let X be covered by a finite $\{U\}$ and let

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of depedent abelian groups on X. If $\{U\}$ is Čech-trivializing for $\mathcal{F}$, then a long exact sequence of Čech-Cohomology groups is induced:

$$\begin{tikzcd}
 & \dots \arrow[r] & \check{H}^{k-1}(\{U\},\mathcal{H}) \arrow[dll] \\
\check{H}^k(\{U\},\mathcal{F}) \arrow[r] & \check{H}^k(\{U\},\mathcal{G}) \arrow[r] & \check{H}^k(\{U\},\mathcal{H}) \arrow[dll] \\
\check{H}^{k+1}(\{U\},\mathcal{F}) \arrow[r] & \dots &
\end{tikzcd}$$

Proof The cover is Čech-trivializing for $\mathcal{F}$, so $H^1(I^k_x,\mathcal{F}) = \bigoplus_{i_0,\dots,i_k} H^1(U_{i_0\dots i_k}(x),\mathcal{F}) = 0$. Using the long exact sequence for Eilenberg-MacLane Cohomology Proposition 1.0.11, this means that for all $x : X$, the sequence

$$0 \to \mathcal{F}^{I^k_x} \to \mathcal{G}^{I^k_x} \to \mathcal{H}^{I^k_x} \to 0$$

is exact, which implies exactness of all sequences:

$$0 \to \check{\mathcal{F}}^k \to \check{\mathcal{G}}^k \to \check{\mathcal{H}}^k \to 0.$$

The cover is Čech-trivializing for $\mathcal{F}$, so using Lemma 1.0.7 and Lemma 1.0.6 we have

$$\begin{aligned}
H^1(X,\check{\mathcal{F}}^k) &= H^1(X, i^k_* i^{k*}\mathcal{F}) \\
&= H^1((x : X) \times (i_0,\dots,i_k) \times U_{i_0\dots i_k}(x), i^{k*}\mathcal{F}) \\
&= \bigoplus_{i_0\dots i_k} H^1(U_{i_0\dots i_k},\mathcal{F}) \\
&= 0
\end{aligned}$$

This implies, by the long exact sequence for non-Čech cohomology, that applying $\prod$ preserves the exactness. So by Remark 2.0.3, we have short exact sequences:

$$0 \to \bigoplus_{i_0,\dots,i_k} \mathcal{F}(U_{i_0\dots i_k}) \to \bigoplus_{i_0,\dots,i_k} \mathcal{G}(U_{i_0\dots i_k}) \to \bigoplus_{i_0,\dots,i_k} \mathcal{H}(U_{i_0\dots i_k}) \to 0$$

which assemble to a short exact sequence of chain complexes. By homological algebra, this induces the desired long exact sequence of the cohomology groups of these complexes. □

A very specific consequence we will need for the proof that Čech cohomology is a universal ∂-functor:

Corollary 2.0.6 Let X have a cover $\{U\}$ with the same properties as in the theorem with respect to all $\mathcal{F}_1,\dots,\mathcal{F}_n$. Then, all higher Čech-Cohomology groups of $\bigoplus_i \mathcal{F}_i$ vanish, if they vanish for all the $\mathcal{F}_i$.

3 Cohomology of affine schemes

Let R be a fixed commutative ring, serving as a base ring for the definitions from the preprint [CCH23], we will now import:

Definition 3.0.1 Let A be an R-algebra.

(a) For $r : R$ let

$$D(r) :\equiv r \text{ is invertible}$$

be the proposition that r has a multiplicative inverse.

(b) A subtype $U : X \to \mathrm{Prop}$ of any type X is open, if for all $x : X$, there merely are $r_1,\dots,r_n$ such that $U(x) = D(r_1) \vee \dots \vee D(r_n)$.

(c) The type

$$\mathrm{Spec}\, A :\equiv \mathrm{Hom}_R(A, R)$$

of R-algebra homomorphisms is called the *spectrum of* A and there is a correspondence with external affine spectra in the Zariski-topos.

(d) A scheme is a type X which is covered by finitely many open affine subtypes. These schemes are expected to correspond to external quasi-compact, quasi-separated schemes, locally of finite presentation.

Definition 3.0.2 Let M be an R-module. M is *weakly quasi-coherent*, if the canonical R-linear map

$$\frac{m}{r^k} \mapsto ((\_ : r \text{ inv}) \mapsto r^{-k}\frac{m}{r^k}) : M_f \to M^{D(r)}$$

is an isomorphism. We denote the type of weakly quasi-coherent R-modules with R-$\mathrm{Mod}_{\mathrm{wqc}}$.

Remark 3.0.3 Let $M : X \to R\text{-}\mathrm{Mod}_{\mathrm{wqc}}$. Then for any $f : X \to R$ there are isomorphisms of R^X-modules

$$M(D(f)) :\equiv \prod_{x:D(f)} M_x = \prod_{x:X} M_{f(x)} = \prod_{x:X} M^{D(f(x))}.$$

Proof [CCH23][Lemma 7.1.4]. □

Theorem 3.0.4
Let $X = \mathrm{Spec}(A)$ be an affine scheme, $M : X \to R\text{-}\mathrm{Mod}_{\mathrm{wqc}}$ a family of weakly quasi-coherent R-modules, and $n > 0$. Then we have

$$H^n(X; M) = 0.$$

Proof We induct on n. The base case $n = 1$ is [CCH23][Theorem 8.3.6]. Thus suppose $n \geq 2$ and that the theorem holds for all $0 < l < n$ and any X, M.

Let $\chi : (x : X) \to K(M_x, n)$ represent a cohomology class. We wish to show $\|\chi = 0\|$, a proposition. We know that $\|\chi(x) = 0\|$ for all x, since $K(M_x, n)$ is connected. By Zariski choice, we obtain a covering $X = \bigcup_{i\in[m]} U_i$, such that $\chi(x) = 0$ for $x \in U_i$, and such that the proposition $x \in U_i$ is standard open for each x, i. For $x : X$, let $I_x :\equiv (i : [m]) \times (x \in U_i)$. Note that I_x is an affine scheme, since affine schemes are closed under finite coproducts.

Since $\chi(x) = 0$ when $x \in U_i$, the image of $\chi(x)$ under the diagonal map $K(M_x, n) \to K(M_x, n)^{I_x}$ is zero. This diagonal map can be factored as $K(M_x, n) \to K(M_x^{I_x}, n) \to K(M_x, n)^{I_x}$, where the first map is induced by the diagonal $\Delta_x : M_x \to M_x^{I_x}$, and the second is given by the equivalence $M_x^{I_x} \simeq \Omega^n(K(M_x, n)^{I_x})$ ((Cite David's preprint?)). We claim that $\chi(x)$ maps to zero already in $K(M_x^{I_x}, n)$. To this end, it suffices to show that $K(M_x^{I_x}, n) \to K(M_x, n)^{I_x}$ is an embedding. Since the domain is connected, it suffices to show that this map becomes an equivalence after applying Ω. So we need to show that the canonical map $K(M_x^{I_x}, n-1) \to K(M_x, n-1)^{I_x}$ is an equivalence. It becomes an equivalence after applying Ω^{n-1}, and the domain is $(n-2)$-connected, so by Whitehead's principle it suffices to show that the codomain is also $(n-2)$-connected. Since $\pi_j(K(M_x, n-1)^{I_x}) = H^{n-1-j}(I_x; M_x)$, it suffices to show that $H^l(I_x; M_x) = 0$ for $0 < l \leq n-1$. This follows from induction hypothesis (using that I_x is an affine scheme).

From this we can conclude that χ maps to zero in $H^n(X; M_x^{I_x})$. Since I_x is merely inhabited, Δ_x is an embedding. Hence we have a short exact sequence $0 \to M_x \to M_x^{I_x} \to \mathrm{coker}\, \Delta_x \to 0$. This induces a long exact sequence on cohomology. One part of this long exact sequence is $H^{n-1}(X; \mathrm{coker}\, \Delta_x) \to H^n(X; M_x) \to H^n(X; M_x^{I_x})$. By inductive hypothesis, $H^{n-1}(X; \mathrm{coker}\, \Delta_x) = 0$ (using that weakly quasi-coherent modules are closed under cokernels of monomorphisms, finite products, and exponentiation with standard opens). Hence $H^n(X; M_x)$ embeds in $H^n(X; M_x^{I_x})$, so χ must already have been zero in $H^n(X; M_x)$, as needed. □

One should be able to follow the above reasoning to show also vanishing of H^1, provided we know that H^0 is right exact.

4 Čech cohomology of a join

Definition 4.0.1 The join $X * Y$ of two types X, Y is given by the following pushout.

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & X * Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Let n be a natural number and $P_1, \ldots, P_n$ types. We define the join $P_1 * \cdots * P_n$ by induction on n, so that it is empty if $n = 0$ and $P_1 * (P_2 * \cdots * P_n)$ if $n \geq 1$. Our goal is to describe a precise sense in which this join is built from the products $\Pi_{i:I} P_i$ where $I \subseteq [n]$ ranges over detachable, inhabited subsets. Note that if P_i are all propositions, then so is the join, with $P_1 * \cdots * P_n = P_1 \vee \cdots \vee P_n$.

Definition 4.0.2 We define a sequence $J_{-1} \to J_0 \to J_1 \to \cdots$ of types. If $n = 0$, we take $J_r = \varnothing$ for all r. If $n > 0$, let $\widehat{J}_r$ be the sequence obtained recursively from the types $P_2, \ldots, P_n$. We take $J_{-1} = \varnothing$ and for $r \geq 0$ define J_r by the following pushout diagram.

$$\begin{tikzcd} P_1 \times \widehat{J}_{r-1} \arrow[r] \arrow[d] & P_1 \arrow[d] \\ \widehat{J}_r \arrow[r] & J_r \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The map $J_r \to J_{r+1}$ is induced by functoriality of pushouts via the following commutative diagram.

$$\begin{tikzcd} \widehat{J}_{r-1} \arrow[d] & P_1 \times \widehat{J}_{r-2} \arrow[l] \arrow[r] \arrow[d] & P_1 \arrow[d] \\ \widehat{J}_r & P_1 \times \widehat{J}_{r-1} \arrow[l] \arrow[r] & P_1 \end{tikzcd}$$

Lemma 4.0.3 For $r \geq n - 1$, the map $J_r \to J_{r+1}$ is an equivalence and $J_r \simeq P_1 * \cdots * P_n$ is the join.

Proof Direct by induction on n. □

Definition 4.0.4 For r a natural number, let $[n]^{(r)}$ denote the type of r-element subsets of $[n]$, and define

$$Z_r :\equiv (I : [n]^{(r)}) \times (i : I) \to P_i.$$

Lemma 4.0.5 For $r \geq 0$, we have a pushout square of the following form.

$$\begin{tikzcd} Z_{r+1} \times S^{r-1} \arrow[r] \arrow[d] & Z_{r+1} \arrow[d] \\ J_{r-1} \arrow[r] & J_r \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

That is, J_r is obtained from J_{r-1} by attaching Z_{r+1}-many r-cells.

Proof We induct on n. For $n = 0$, Z_{r+1} is empty and so there is nothing to prove. For $r = 0$ the conclusion is also clear. Thus suppose $n > 0$, $r > 0$ and that the lemma holds for the sequence $P_2, \ldots, P_n$. Consider the following 3-by-3-diagram, with the pushouts of the rows and columns listed at the bottom and to the right.

$$\begin{tikzcd} P_1 & P_1 \times \widehat{Z}_r \arrow[l] \arrow[r, equal] & P_1 \times \widehat{Z}_r & P_1 \\ P_1 \times \widehat{J}_{r-2} \arrow[u] \arrow[d] & P_1 \times \widehat{Z}_r \times S^{r-2} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & P_1 \times \widehat{Z}_r \arrow[u, equal] \arrow[d] & P_1 \times \widehat{J}_{r-1} \arrow[u] \arrow[d] \\ \widehat{J}_{r-1} & \widehat{Z}_{r+1} \times S^{r-1} + P_1 \times \widehat{Z}_r \arrow[l] \arrow[r] & \widehat{Z}_{r+1} + P_1 \times \widehat{Z}_r & \widehat{J}_r \\ J_{r-1} & Z_{r+1} \times S^{r-1} \arrow[l] \arrow[r] & Z_{r+1} & J_r \end{tikzcd}$$

The maps in this diagram are all guessable, and the commutativity of each of the four squares is direct. We explain how to compute the pushout of each row and column. The pushout of the first row is P_1, since the pushout of any equivalence is an equivalence. The pushout of the second row is $P_1 \times \widehat{J}_{r-1}$, by inductive hypothesis and using that $P_1 \times -$ preserves pushouts. The pushout of the third row is $\widehat{J}_r$, again using inductive hypothesis as well as the observation that the $P_1 \times \widehat{Z}_r$-terms do not affect the pushout.

The pushout of the first column is J_{r-1} by definition. To compute the pushout of the second column, we observe that the $\widehat{Z}_{r+1} \times S^{r-1}$-term does not interact with the rest of the column, that the suspension of S^{r-2} is S^{r-1}, and that $P_1 \times \widehat{Z}_r \times -$ preserves pushouts. All together, this shows that the pushout is $\widehat{Z}_{r+1} \times S^{r-1} + P_1 \times \widehat{Z}_r \times S^{r-1}$, i.e. $(\widehat{Z}_{r+1} + P_1 \times \widehat{Z}_r) \times S^{r-1}$, i.e. $Z_{r+1} \times S^{r-1}$. Finally, the third pushout is $\widehat{Z}_{r+1} + P_1 \times \widehat{Z}_r$ since the pushout of an equivalence is an equivalence, i.e. Z_{r+1}.

The 3×3-lemma tells us that the the pushout of row-wise pushouts is equivalent to the pushout of column-wise pushouts. That is, J_r is a pushout of the desired form. (Here one should be careful to check that the maps are the ones we expect.) □

Now let X be a type, n a natural number, and P_i a type family over X for each $i : [n]$. For any $x : X$, $P_1(x), \ldots, P_n(x)$ is simply a list of types, to which we may apply Lemma 4.0.5. Taking sigma over $x : X$ preserves pushouts (since it is a left adjoint), so we obtain the following pushout square for each $r \geq 0$.

$$\begin{tikzcd}
(x : X) \times Z_{r+1}(x) \times S^{r-1} \arrow[r] \arrow[d] & (x : X) \times Z_{r+1}(x) \arrow[d] \\
(x : X) \times J_{r-1}(x) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & (x : X) \times J_r(x)
\end{tikzcd} \tag{1}$$

We could now use Mayer–Vietoris, but instead let us consider the following lemma which explains how to compute the cohomology of such a pushout. It is in the spirit of cellular cohomology.

Lemma 4.0.6 Let $n \geq 0$ a natural number, and suppose given the following pushout square.

$$\begin{tikzcd}
I \times S^n \arrow[r] \arrow[d] & I \arrow[d] \\
C \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & C'
\end{tikzcd}$$

Then for any parametrised spectrum A over C' we have the following fibre sequence of spectra, where A^C denotes the cohomology spectrum of C with coefficients in A:

$$A^{C'} \to A^C \to \Omega^n A^I.$$

Proof By the universal property of A^-, it turns colimits into limits, so we have the following pullback square in spectra.

$$\begin{tikzcd}
A^{C'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A^I \arrow[d] \\
A^C \arrow[r] & A^{I \times S^n}
\end{tikzcd}$$

We compute the bottom-right term:

$$A^{I \times S^n} \simeq (A^{S^n})^I \simeq (A \oplus \Omega^n A)^I \simeq A^I \oplus \Omega^n A^I,$$

using that $(S^n \to B) \simeq B \times \Omega^n B$ for any homogeneous pointed type B. Here $\oplus$ denotes biproduct of spectra. It can be seen that the map $A^I \to A^{I \times S^n}$ corresponds to the left inclusion into this direct sum. We can then paste this pullback diagram with

$$\begin{tikzcd}
A^I \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\
A^I \oplus \Omega^n A \arrow[r] & \Omega^n A^I
\end{tikzcd}$$

to get the desired fibre sequence. □

Now suppose M is a family of abelian groups over X, and we are interested in cohomology with coefficients in M. Note that $(x : X) \times Z_{r+1}$ is a finite coproduct, and cohomology is additive in those, so we have

$$H^l((x : X) \times Z_{r+1}(x)) \cong \bigoplus_{I:[n]^{(r+1)}} H^l((x : X) \times (i : I) \to P_i(x)).$$

Let us write C_{r+1} for this group in the case where $l = 0$.

Lemma 4.0.7 If P_i is acyclic with regard to M in the sense that $H^l((x : X) \times (i : I) \to P_i(x)) = 0$ for $l > 0$ and any $r \geq 1$, $I : [n]^{(r)}$, then we have the following description of $H^l_r :\equiv H^l((x : X) \times J_r(x))$:

$$H^l_r \cong H^l_{r-1}$$

for $l < r - 1$ (where the map is induced by the map $J_{r-1}(x) \to J_r(x)$),

$$H^{r-1}_r \cong \ker \delta$$

and

$$H^r_r \cong \operatorname{coker} \delta$$

where $\delta : H^{r-1}_{r-1} \to C_{r+1}$ is induced by the attaching map $Z_{r+1}(x) \times S^{r-1} \to J_r(x)$, and

$$H^l_r \cong 0$$

for $l > r$.

Proof We induct on r. For $r = -1$ there is nothing to prove. For $r \geq 0$ we start with the pushout square (1). To this we apply Lemma 4.0.6, obtaining a fibre sequence of cohomology spectra. Consider the associated long exact sequence on homotopy groups. By assumption, one of the spectra in our fibre sequence has its cohomology concentrated in degree $r-1$, where it is C_{r+1}, leading to the desired result. $\square$

From Lemma 4.0.7, we have that H^r_r is a quotient of C_{r+1} for all r. In particular, the map $\delta : H^{r-1}_{r-1} \to C_{r+1}$ appearing in the statement of Lemma 4.0.7 composes with the quotient map to give a map $C_r \to C_{r+1}$. We expect this map to be the usual boundary map appearing in the Čech complex. Modulo this gap, we arrive at our main result.

Theorem 4.0.8
For any $P_1, \dots, P_n$ which are acyclic with regard to M, the cohomology groups

$$H^l((x : X) \times P_1(x) * \cdots * P_n(x); M)$$

of the fibrewise join with coefficients in M agree with the cohomology of the Čech complex, i.e. the kernel of the boundary map $C_{l+1} \to C_{l+2}$ modulo the image of the boundary map $C_l \to C_{l+1}$.

Proof Combining Lemmas 4.0.3 and 4.0.7. (Modulo the gap about describing the boundary map $C_r \to C_{r+1}$.) $\square$

Note that if $P_i(x)$ are propositions, then the fibrewise join is simply the union of subtypes, and the types $(x : X) \times (i : I) \to P_i(x)$ appearing in the Čech complex are simply intersections of subtypes.

5 ∂-Functors

Cohomology has the universal property of being a *universal ∂-functor*. In this section, we will construct a tool for proving this in some particular situations, both for the cohomology defined using Eilenberg-MacLane spaces and Čech cohomology.

The following definition, from ([Gro, p. 2.1]) and originally from ([Gro57]), is specialized to our needs. Grothendieck makes a definition for additive functors from an abelian category to a preadditive category. We will only need the theory for functors from certain subcategories of dependent R-modules over a fixed type to abelian groups. Also, some arguments are a lot more convenient when we can use elements of modules instead of abstract categorical language. Therefore, we will state our definitions and results only for this particular situation.

Let R be a fixed commutative ring and $\mathcal{A}$ be a subcategory of the category of dependent R-modules over a fixed type X, which is closed under finite direct sums.

Definition 5.0.1 An *(l-truncated) ∂-functor* is a collection of additve[1] functors $T^i : \mathcal{A} \to \mathrm{Ab}$, where $0 \le i < l$ with $l \in \mathbb{N} \cup \{\infty\}$, together with a collection of connecting morphisms $\partial_{S,i}$ for any short exact sequence S and $0 \le i \le l$, subject to the following conditions:

(a) Let $\mathcal{S}$ be a short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathcal{A}$. Applying the T^i yields a complex, together with connecting morphisms $(\partial_{\mathcal{S},i})_{0 \le i < l-1}$:

$$0 \longrightarrow T^0(A') \longrightarrow T^0(A) \longrightarrow T^0(A'') \xrightarrow{\partial_{\mathcal{S},0}} T^1(A') \longrightarrow T^1(A) \longrightarrow \dots$$

(b) For any homomorphism to a second short exact sequence

$$0 \to B' \to B \to B'' \to 0$$

and any $i < l-1$ the corresponding square commutes:

$$\begin{array}{ccc} T^i(A'') & \xrightarrow{\partial} & T^{i+1}(A') \\ \downarrow & & \downarrow \\ T^i(B'') & \xrightarrow{\partial} & T^{i+1}(B') \end{array}$$

Definition 5.0.2 Let $l, k : \mathbb{N}$. The l-th *truncation* of a $(l+k)$-truncated ∂-functor T is just the restriction of $(T^i)_{i<l+k}$ to $(T^i)_{i<l}$, together with a restriction of the ∂-maps and we denote the l-th truncation with $T^{\le l}$.

Definition 5.0.3 Let T and T' be ∂-functors defined for the same indices.

A *morphism of ∂-functors* $f : T \to T'$ is given by a natural transformation $f^i : T^i \to T'^i$ for each valid i, such that for any short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

the following square commutes:

$$\begin{array}{ccc} T^i(A'') & \xrightarrow{\partial} & T^{i+1}(A') \\ \big\downarrow{\scriptstyle f^i_{A''}} & & \big\downarrow{\scriptstyle f^{i+1}_{A'}} \\ T'^i(A'') & \xrightarrow{\partial} & T'^{i+1}(A') \end{array}$$

Definition 5.0.4 A ∂-functor T is called *exact*, if all values are exact complexes.

Definition 5.0.5 A ∂-functor T is called *universal*, if for any T', defined for the same indices, any natural transformation $f^0 : T^0 \to T'^0$ extends uniquely to a morphism of ∂-functors $f : T \to T'$.

To prove that some ∂-functor has this universal property, we will *extend* morphisms of ∂-functors, level by level. By observing the diagram in the proof of the lemma below, one can see that this is possible using exact sequences with the property, that some particular element is zero in their middle term. This property will appear often enough to deserve a name:

Definition 5.0.6 Let T be a ∂-functor, i a valid index, $A : \mathcal{A}$ and $\chi : T^i(A)$. We say that a short exact sequence $\mathcal{S} = A \to R \to S$ *resolves* χ, if χ is mapped to zero in $T^i(R)$.

In the classical approach with injective resolutions, for a fixed $A : \mathcal{A}$ all elements of $T^i(A)$ for all $i > 0$ would be resolved. For our examples, where we can resolve elements of $H^i(X, A)$, we will only be able to *merely* resolve one $\chi : H^i(X, A)$ at a time. So resolving all elements at once with the same construction, would require some form of choice.

We will now show, how a short exact sequence resolving an element might be of use to extend morphisms of ∂-functors.

[1]The zero object and binary direct sums are preserved.

Lemma 5.0.7 Let $l \geq i > 0$ and T be an exact, l-truncated ∂-functor, $\mathcal{S} = A \to R_\chi \to S_\chi$ a short exact sequence in $\mathcal{A}$ and $\chi : T^i(A)$

$$A \xrightarrow{r_\chi} R_\chi \xrightarrow{s_\chi} S_\chi$$

that resolves χ, i.e. such that $T^i(r_\chi)(\chi) = 0$. For an l-truncated ∂-functor T' and any morphism of $(i-1)$-truncated ∂-functors $f : T^{\leq(i-1)} \to T'^{\leq(i-1)}$, there is a unique

$$\mathrm{ext}(f, \chi, \mathcal{S}) : T'^i(A)$$

such that for any $x : T^{i-1}(S_\chi)$ with $\partial_{T,S,i-1}(x) = |\chi|$ we have $\partial_{T,S,i-1}(f^{i-1}(x)) = \mathrm{ext}(f, \chi, \mathcal{S})$.

Proof The following diagram commutes:

$$\begin{tikzcd}
T^{i-1}(A) \arrow[r] \arrow[d, "f^{i-1}"] & T^{i-1}(R_\chi) \arrow[r] \arrow[d, "f^{i-1}"] & T^{i-1}(S_\chi) \arrow[r, "\partial"] \arrow[d, "f^{i-1}"] & T^i(A) \arrow[r, "r_\chi^*"] & T^i(R_\chi)\dots \\
T'^{i-1}(A) \arrow[r] & T'^{i-1}(R_\chi) \arrow[r, "s_\chi^*"] & T'^{i-1}(S_\chi) \arrow[r, "\partial"] & T'^i(A) \arrow[r] & T'^i(R_\chi)\dots
\end{tikzcd}$$

The upper row is exact and the lower row is a complex.

Let $E(\chi, \mathcal{S})$ be the type of all possible values of f^i in $T'^i(A)$, with which the dependent sum over all $y : T'^i(A)$ such that there merely is $x : T^{i-1}(S_\chi)$ with $\partial(x) = |\chi|$ and $\partial(f^{i-1}(x)) = y$. Then $E(\chi, \mathcal{S})$ is inhabited, since $r_\chi(|\chi|) = 0$ and by exactness, there has to be a mere preimage under ∂. So we need to show, that $E(\chi, \mathcal{S})$ is a proposition.

Let $x : T^{i-1}(S_\chi)$ such that $\partial(x) = |\chi|$. Then any other element with this property will be of the form $x + k$, with k in the kernel of ∂. Any k like that, has a mere preimage $k' : T^{i-1}(R_\chi)$ and since the lower row is a complex, we have $\partial(s_\chi^*(f^{i-1}(k'))) = 0$.

So for any extension $y : T'^i(A)$ we have

$$\begin{aligned}
y &= \partial(f^{i-1}(x + k)) \\
&= \partial(f^{i-1}(x)) + \partial(f^{i-1}(k)) \\
&= \partial(f^{i-1}(x)) + \partial(s_\chi^*(f^{i-1}(k'))) \\
&= \partial(f^{i-1}(x))
\end{aligned}$$

This means we can define $\mathrm{ext}(f, \chi, \mathcal{S})$ to be the unique element of $E(\chi, \mathcal{S})$. $\square$

While this shows, that existence of these special short exact sequences is enough to extend a *map* from one truncation level to the next, it is not clear, that an extension constructed in this way, is actually a morphism of truncated ∂-functors.

It is also unclear, if the construction even yields a well-defined map, independent of the short exact sequence we chose in the construction. A solution to these problems is essentially given by requiring some "functoriality" of the short exact sequences we will use (definition 5.0.9) and the following naturality result:

Lemma 5.0.8 Let T be an exact ∂-functor. Let $\chi : T^i(A)$ and

$$\begin{tikzcd}
A \arrow[r, "r_\chi"] \arrow[d, "\varphi"] & R_\chi \arrow[r] \arrow[d, "\varphi_R"] & S_\chi \arrow[d, "\varphi_S"] \\
A' \arrow[r] & R_{\varphi(\chi)} \arrow[r] & S_{\varphi(\chi)}
\end{tikzcd}$$

be a morphism of short exact sequences $\mathcal{S}_\chi$ and $\mathcal{S}_{\varphi(\chi)}$ in $\mathcal{A}$, where $T^i(r_\chi)(\chi) = 0$. Then, for the construction from lemma 5.0.7, we have the following commutativity:

$$T^i(\varphi)(\mathrm{ext}(f, \chi, \mathcal{S}_\chi)) = \mathrm{ext}(f, \varphi(\chi), \mathcal{S}_{\varphi(\chi)})$$

Proof (of lemma 5.0.8) Let T' be another ∂-functor and $f : T^{\leq i-1} \to T'^{\leq i-1}$. Apply the ∂-Functors T and T' to the morphism of short exact sequences, to get the following diagram:

$$\begin{tikzcd}[column sep=small]
T^{i-1}(R_\chi) \arrow[rr] \arrow[dr] \arrow[dd, "f^{i-1}"] & & T^{i-1}(S_\chi) \arrow[rr] \arrow[dr] \arrow[dd, "f^{i-1}" near start] & & T^i(A) \arrow[rr] \arrow[dr, "\varphi^*"] & & T^i(R_\chi)\dots \arrow[dr] & \\
& T^{i-1}(R_{\varphi(\chi)}) \arrow[rr] \arrow[dd, "f^{i-1}" near start] & & T^{i-1}(S_{\varphi(\chi)}) \arrow[rr] \arrow[dd, "f^{i-1}" near start] & & T^i(A') \arrow[rr] & & T^i(R_{\varphi(\chi)})\dots \\
T'^{i-1}(R_\chi) \arrow[rr] & & T'^{i-1}(S_\chi) \arrow[rr] \arrow[dr] & & T'^i(A) \arrow[rr] \arrow[dr, "T'^i(\varphi)"] & & T'^i(R_\chi)\dots \arrow[dr] & \\
& T'^{i-1}(R_{\varphi(\chi)}) \arrow[rr] & & T'^{i-1}(S_{\varphi(\chi)}) \arrow[rr] & & T'^i(A') \arrow[rr] & & T'^i(R_{\varphi(\chi)})\dots
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
a \arrow[rr, mapsto] \arrow[dr, mapsto] \arrow[dd, mapsto] & & \chi \arrow[rr, mapsto] \arrow[dr, mapsto] & & 0 \arrow[dr, mapsto] & \\
& a' \arrow[rr, mapsto] \arrow[dd, mapsto] & & \varphi(\chi) \arrow[rr, mapsto] & & 0 \\
b \arrow[rr, mapsto] \arrow[dr, mapsto] & & \mathrm{ext}(\chi, R_\chi) \arrow[dr, mapsto] & & & \\
& b' \arrow[rr, mapsto] & & ? & &
\end{tikzcd}$$

From exactness of the upper sequence, we get that there is a preimage a of χ. Let a' denote the image of a in $T^{i-1}(S_{\varphi(\chi)})$, then a' will be a preimage of $\varphi(\chi)$ in the parallel sequence by commutativity. That means, that b', the image of a' in the lower sequence, will be mapped to $\mathrm{ext}(f, \varphi(\chi), \mathcal{S}_{\varphi(\chi)})$, but by commutativity, $\mathrm{ext}(f, \chi, \mathcal{S}_\chi)$ will be mapped to the same thing by $T^i(\varphi)$. So:

$$T^i(\varphi)(\mathrm{ext}(f, \chi, \mathcal{S}_\chi)) = \mathrm{ext}(f, \varphi(\chi), \mathcal{S}_{\varphi(\chi)}) \qquad \square$$

We summarize the exact condition we found useful to prove universality of ∂-functors, together with the existence of enough "good" short exact sequences in the following definition.

(The following is about to be replaced with a more specialized but easier notion, focusing on trivializing covers instead of resolving sequences. One good thing about asking for trivializing covers should be that the big diagram below will only appear in one proof instead of a couple of places.)

Definition 5.0.9 Let T be a ∂-functor. We say that $\mathcal{A}$ *has local resolutions for* T, if

(i) For any $i > 0$, $A : \mathcal{A}$ and $\chi : T^i(A)$ there merely is a short exact sequence:

$$0 \longrightarrow A \xrightarrow{m_\chi} M_\chi \longrightarrow C_\chi \longrightarrow 0$$

resolving χ, i.e. such that $T^i(m_\chi)(\chi) = 0$.

(ii) For any short exact sequence $\mathcal{S} = A \to R \to S$ resolving χ and any morphism $\varphi : A \to B$, there is a zig–zag of short exact sequences resolving χ or, respectively $\varphi(\chi)$, of the following shape:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & R \arrow[r] \arrow[d] & S \arrow[d] \\
A \arrow[r] \arrow[d, equal] & R_1 \arrow[r] & S_1 \\
\vdots & \vdots \arrow[u] & \vdots \arrow[u] \\
A \arrow[r] \arrow[d, "\varphi"] & R_l \arrow[r] \arrow[d] & S_l \arrow[d] \\
B \arrow[r] \arrow[d, equal] & M_{\varphi,\mathcal{S}} \arrow[r] & C_{\varphi,\mathcal{S}} \\
B \arrow[r, "m_{\varphi(\chi)}"] & M_{\varphi(\chi)} \arrow[r] \arrow[u] & C_{\varphi(\chi)} \arrow[u]
\end{tikzcd}$$

The following is provable by a constructive adaption of Prop 2.2.1 in [Gro]:

Theorem 5.0.10
Let X be a type. An exact ∂-functor $(T^i)_{i<l}$ from $\mathcal{A}$ to Ab is universal, if $\mathcal{A}$ has local resolutions for T.

Proof (of theorem 5.0.10) To extend a given morphism f^0, we will construct $f^i : T^i \to T'^i$ by induction on i for $0 < i < l$. So let T' be a ∂-functor and assume, we already have a morphism for $i-1$ and lower indices. We start by constructing a group homomorphism $f^i : T^i(A) \to T'^i(A)$ for arbitrary $A : \mathcal{A}$.

By lemma 5.0.7, we merely get $f^i(\chi) :\equiv \mathrm{ext}(f, \chi, \mathcal{S}_\chi)$, for each $\chi : T^i(A)$ and their merely given local resolutions $\mathcal{S}_\chi$. To see that this yields an actual map, we have to check that the values $\mathrm{ext}(f, \chi, \mathcal{S}_\chi)$ are independent of the short exact sequence $\mathcal{S}_\chi$. For any other short exact sequence $\mathcal{S}' = A \to R_\chi \to S_\chi$ that resolves χ, we get a zig-zag by our requirement on local resolutions:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & R_\chi \arrow[r] \arrow[d] & S_\chi \arrow[d] \\
A \arrow[r] \arrow[d, equal] & R_1 \arrow[r] & S_1 \\
\vdots & \vdots \arrow[u] & \vdots \arrow[u] \\
A \arrow[r] \arrow[d, "\mathrm{id}"] & R_l \arrow[r] \arrow[d] & S_l \arrow[d] \\
A \arrow[r] \arrow[d, equal] & M_{\mathrm{id},\mathcal{S}'} \arrow[r] & C_{\mathrm{id},\mathcal{S}'} \\
A \arrow[r, "m_\chi"] & M_\chi \arrow[r] \arrow[u] & C_\chi \arrow[u]
\end{tikzcd}$$

Applying lemma 5.0.8 to any of these morphisms $\mathcal{S} \to \mathcal{S}'$ of exact sequences gives us:

$$\mathrm{ext}(f, \chi, \mathcal{S}) = T^i(\mathrm{id})(\mathrm{ext}(f, \chi, \mathcal{S}')) = \mathrm{ext}(f, \chi, \mathcal{S}')$$

So we have a well-defined map $f^i : T^i(A) \to T'^i(A)$. We will show that it is a homomorphism of groups. First, note that $f^i(0) = 0$, because 0 has the identity as a local resolution, i.e. the sequence $0 \to A \to A \to 0 \to 0$.

Now let $\xi, \chi, \xi + \chi : T^i(A)$. We need to show that $f^i(\xi) + f^i(\chi) = f^i(\xi + \chi)$. By additivity of the T^i, we can "resolve" these three elements at once, applying our construction to $(\xi, \chi, \xi + \chi) : T^i(A \oplus A \oplus A)$. Again by lemma 5.0.8 using the inclusions $A \to A \oplus A \oplus A$, we get $f^i(\xi, \chi, \xi + \chi) = (f^i(\xi), f^i(\chi), f^i(\xi + \chi))$. By using lemma 5.0.8 on the map $a :\equiv (x, y, z) \mapsto x + y - z : A \oplus A \oplus A \to A$, we get:

$$\begin{aligned}
f^i(\xi) + f^i(\chi) - f^i(\xi + \chi) &= T^i(a)((f^i(\xi), f^i(\chi), f^i(\xi + \chi))) \\
&= f^i(a(\xi, \chi, \xi + \chi)) \\
&= f^i(0) \\
&= 0.
\end{aligned}$$

This shows that f^i is as homomorphism.

Let $\mathcal{S} = A \to B \to C$ be an arbitrary exact sequence. To see that f^i commutes with the connecting morphism $\partial_{\mathcal{S},i-1}$, let $x : T^{i-1}(C)$ and χ be the image of x in $T^i(A)$. By exactness, χ will be mapped to 0 in $T^i(B)$, so $\mathcal{S}$ resolves χ and therefore, the desired commutativity follows from the well-definedness proof for f^i.

The only thing left to show is that f^i is a natural transformation $T^i \to T'^i$. Let $\varphi : A \to B$ and $\chi : T^i(A)$. By our definition of local resolutions, there is a zig-zag:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & R_\chi \arrow[r] \arrow[d] & S_\chi \arrow[d] \\
A \arrow[r] \arrow[d, equal] & R_1 \arrow[r] & S_1 \\
\vdots & \vdots \arrow[u] & \vdots \arrow[u] \\
A \arrow[r] \arrow[d, "\varphi"] & R_l \arrow[r] \arrow[d] & S_l \arrow[d] \\
B \arrow[r] \arrow[d, equal] & M_{\varphi,\mathcal{S}'} \arrow[r] & C_{\varphi,\mathcal{S}'} \\
B \arrow[r, "m_{\varphi(\chi)}"] & M_{\varphi(\chi)} \arrow[r] \arrow[u] & C_{\varphi(\chi)} \arrow[u]
\end{tikzcd}
$$

and therefore by applying lemma 5.0.8 to the but last rectangle:

$$
\begin{aligned}
T^i(\varphi)(f^i(\chi)) &= T^i(\varphi)(\mathrm{ext}(f, \chi, \mathcal{S}_l)) \\
&= \mathrm{ext}(f, \varphi(\chi), \mathcal{M}) \\
&= f^i(\varphi(\chi)).
\end{aligned}
$$

□

6 Construction of local resolutions

(META: This section is somewhat incomplete, but expected to work out with high level of confidence.)

6.1 General local resolutions

(TODO: For the vanishing result and join-based chech cohomology, we also use what is explained below. That should somehow be consolidated/brought into the right order.)

Among 1-truncated ∂-functors, H^0, H^1, i.e. the zeroth and first cohomology groups defined in terms of Eilenberg-MacLane spaces, will always be universal. We will show this, by constructing local resolutions for (H^0, H^1) in the sense of definition 5.0.9. The construction will follow a general pattern, which we will also use in the following sections for all other resolutions. What follows, is an explanation in more classical terms – a reader not familiar with those, can safely skip that explanation, since the construction we use, will be quite simple and can be understood without the classical notions of torsors or fibre bundles.

Let X be a type and $A : X \to \mathrm{Ab}$. An element $\chi : H^1(X, A)$ can be merely represented by a map $T : (x : X) \to K(A_x, 1)$. If we use the particular implementation of the deloopings $K(A_x, 1)$ as A_x-torsors, T_x will be a set with an action and it is natural to view T as a bundle over X. Let us relax the usual notion of fibre bundle a bit, to also admit the case of our A-torsors, i.e. let the following be the type of A-fibre bundles over X:

$$\sum_{T : X \to \mathcal{U}} \|T_x = A_x\|$$

Then, A-torsors will in particular also be A-fibre bundles.

A canonical trivialization for fibre bundles with constant prescribed fiber is given in ([Che22])[Definition 4.9, Definition 4.11] – but this works for the more general notion as well. The canonical trivialization is given by

$$V_T :\equiv \sum_{x:X} T_x = A_x$$

Then, from the definition above, we get that $\pi_1 : V_T \to X$ is surjective and the second projection will give a trivialization witness for the pullback of T along π_1.

Now, since the local resolution has to be something over X, we have to push-forward the construction above. But that really just means we have to take a dependent product instead of a sum. So the resulting dependent group is just:

$$x \mapsto A_x^{(T_x = *)} : X \to \mathrm{Ab}$$

The canonical map $A_x \to A_x^{(T_x=*)}$ maps an element $a_x : A_x$ to the function $p \mapsto a_x$. Since $T_x = *$, this map will be an embedding.

For a more general $T : (x : X) \to K(A_x, l)$, the type $\|T_x = *\|_0$ will be trivial if $l > 1$ and therefore $A_x \to A_x^{(T_x=*)}$ will be an equivalence, since A_x is 0-truncated. This means the same construction will *not* work for cohomology groups above degree 1, with coefficients in Eilenberg-MacLane spaces. Another way to phrase the problem is, that the spectrum $K(A_x, \_)^{(T_x=*)}$ fails to be an Eilenberg-MacLane spectrum. This happens, if and only if, $T_x = *$ has non-trivial cohomology. So one thing we can use to resolve higher cohomology classes, are covers of X with cohomologically trivial fibers, which we will do in the next section. Now we will show, how we can use the general construction for degree 1 to get all requirements of definition 5.0.9:

Lemma 6.1.1 Let $A : X \to \mathrm{Ab}$ and $\chi : H^1(X, A)$. Then there merely is a short exact sequence $\mathcal{S}_\chi$:

$$0 \longrightarrow A \xrightarrow{m_\chi} M_\chi \longrightarrow C_\chi \longrightarrow 0$$

such that for any other short exact sequence $\mathcal{S} = A \to R \to S$ such that χ is mapped to zero in $H^1(X, R)$ and any morphism $f : A \to B$ we have a zig-zag:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & R \arrow[r] \arrow[d] & S \arrow[d] \\
B \arrow[r] \arrow[d, equal] & M_{f,\mathcal{S}} \arrow[r] & C_{f,\mathcal{S}} \\
B \arrow[r, "m_{f(\chi)}"] & M_{f(\chi)} \arrow[r] \arrow[u] & C_{f(\chi)} \arrow[u]
\end{tikzcd}$$

Proof As explained above, we merely have $T : (x : X) \to K(A_x, 1)$ with $|T|_0 = \chi$ and the definition $M_\chi :\equiv ((x : X) \mapsto A_x^{(T_x=*)}$ together with the diagonal map $(a \mapsto (x, p \mapsto a_x)) : A \to M_\chi$, gives a monomorphism $A \to M_\chi$. So we can take the cokernel, to a short exact sequence as required. $m_\chi^*(\chi)$ is zero by $x \mapsto \mathrm{id}_{(T_x=*)} : (x : X) \to T_x = *$. Note that this construction is well-defined in the sense that for another T' with $|T'| = \chi$, we merely have $\|T = T'\|$ and therefire an isomorphism between the resulting sequences.

Now, let $\mathcal{S} = A \to R \to S$ be a short exact sequence such that χ is mapped to zero in $H^1(X, R)$ and $\varphi : A \to B$ any morphism. For $T : (x : X) \to K(A_x, 1)$ with $|T| = \chi$ and $T' :\equiv (x \mapsto K(\varphi_x, 1)(T_x))$ a zig-zag can be constructed, whose maps we will describe below the diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & R \arrow[r] \arrow[d, "\Delta"] & S \arrow[d, dashed] \\
A \arrow[r, "\varphi\Delta"] \arrow[d, equal] & (x \mapsto R_x^{T_x=*}) \arrow[r] & \mathrm{coker}(\varphi\Delta) \\
A \arrow[r, "\Delta"] \arrow[d, "\varphi"] & (x \mapsto A_x^{T_x=*}) \arrow[r] \arrow[u] \arrow[d] & \mathrm{coker}(\Delta) \arrow[u, dashed] \arrow[d, dashed] \\
B \arrow[r, "\Delta'"] \arrow[d, equal] & (x \mapsto B_x^{T_x=*}) \arrow[r] & \mathrm{coker}(\Delta') \\
B \arrow[r, "\Delta''"] & (x \mapsto B_x^{T'_x=*}) \arrow[r] \arrow[u] & \mathrm{coker}(\Delta'') \arrow[u, dashed]
\end{tikzcd}$$

The maps in the middle column are all given by postcomposition with given maps, except for the last map, which is given by precomposition with a map $T_x = * \to T'_x = *$ given by using the pointed map $K(\varphi_x, 1)$. All maps in the right column, are then induced by the universal property of cokernels. As noted above, the last row is isomorphic to any $\mathcal{S}_\chi$, so the zig-zag does indeed satisfy the specification from definition 5.0.9. $\square$

Theorem 6.1.2
$H^{\leq 1}$ is a 1-truncated, universal ∂-functor.

Proof By lemma 6.1.1 we have local resolutions for $H^{\leq 1}$, we can apply theorem 5.0.10. $\square$

6.2 Local resolutions for schemes

Definition 6.2.1 Let X, Y be schemes.

(a) For $M : Y \to R\text{-Mod}$ and $f : X \to Y$ let $f^*M :\equiv (x : X) \mapsto M_{f(x)}$.

(b) For $M : X \to R\text{-Mod}$ and $f : X \to Y$ let $f_*M :\equiv (y : Y) \mapsto \prod_{x:\mathrm{fib}_f(y)} M_{\pi_1(x)}$.

Both operations preserve weakly quasi-coherent modules by [CCH23][Theorem 9.1.11]. As defined in [CCH23], a *scheme* is a type X, such that there merely is an open cover by affine schemes $U_i = \operatorname{Spec} A_i$. As shown in theorem 3.0.4, higher cohomology with coefficients in weakly quasi-coherent modules is trivial on affine schemes. So we know that, for a general scheme, cohomology will be locally trivial. A *separated scheme*, is defined in [CCH23], as a scheme where equality of points is a *closed* proposition – we will not explain that here and only mention that examples include projective and affine schemes. The consequence of relevance here, is that for a separated scheme, intersections of affine opens are affine. This means the open affines form an acyclic cover:

Remark 6.2.2 For a separated scheme X and $M : X \to R\text{-Mod}_{wqc}$, any open affine cover $(U_i)_{i:I}$ is *acyclic*, i.e[2].

$$H^k(U_{i_0 \cdots l}, M) = 0 \quad \forall l > 0, k > 0 \text{ and } i_0, \ldots, i_l : I.$$

We will use these covers in a way similar to the last section. For a cover $(U_i)_{i:I}$, we can view as a map from the coproduct $u : \coprod_i U_i \to X$. Then pullback along u trivializes all higher cohomology of X with values in $M : X \to R\text{-Mod}_{wqc}$ and we can take the push-forward again to get a candidate for a sequence that resolves higher cohomology classes.

Remark 6.2.3 For a separated scheme X and $M : X \to R\text{-Mod}_{wqc}$. Then, for any finite affine open cover $(U_i)_{i:I}$, $x : X$ and $u : \coprod_i U_i \to X$, the R-linear map of weakly quasi-coherent R-modules

$$\Delta :\equiv m_x \mapsto (\_ \mapsto m_x) : M_x \to M_x^{\mathrm{fib}_u(x)}$$

is an embedding and resolves any $\chi : H^k(X, M)$ with $k > 0$.

Proof $(U_i)_{i:I}$ is a cover, so $\mathrm{fib}_u(x)$ is inhabited and therefore Δ is an embedding. For the resolving-property, we will use that $\mathrm{fib}_u(x)$ is affine. To see that, we compute the fiber as an iterated pullback, starting with an inclusion $\iota_i : U_i \to X$ such that $U_i(x)$:

$$\begin{tikzcd}
\mathrm{fib}_u(x) \arrow[r] \arrow[d] & \coprod_j U_i \cap U_j \arrow[r] \arrow[d] & \coprod_i U_i \arrow[d, "u"] \\
1 \arrow[r] & U_i \arrow[r] & X
\end{tikzcd}$$

The right pullback, $\coprod_j U_i \cap U_j$, is affine, since it is a finite coproduct of the affine schemes $U_i \cap U_j$. The left square is a pullback by pasting, and as a pullback of affine schemes, $\mathrm{fib}_u(x)$ is affine.

So we know that for any $k > 0$, $H^k(\mathrm{fib}_u(x), M_x) = 0$. Equivalently, the latter means $\mathrm{fib}_u(x) \to K(M_x, k)$ is connected. By the uniqueness of connected deloopings, this means that

$$K(M_x, k)^{\mathrm{fib}_u(x)} = K(M_x^{\mathrm{fib}_u(x)}, k).$$

And connectedness of the latter means that all higher cohomology classes are resolved. $\square$

Theorem 6.2.4
For a separated scheme X and $M : X \to R\text{-Mod}_{wqc}$. Then the H^k, for $k \in \mathbb{N}$ form a universal ∂-functor with domain $X \to R\text{-Mod}_{wqc}$.

Proof The resolving sequences can be constructed from the monomorphism in remark 6.2.3 by taking cokernels, which are proven to be weakly quasi-coherent in [CCH23]. The second property of local resolutions can be proven analogous to the last section. $\square$

6.3 Local resolutions for Čech-Cohomology

Let X be a fixed set and $\{U\} = (U_0, \ldots, U_n)$ a fixed cover of X. If $\{U\}$ is acyclic, then the Čech Cohomology of weakly quasi-coherent modules on X will be a universal ∂-functor.

The local resolutions can be constructed using the *Čech-sheaf construction*.

[2] Using notation from definition 2.0.1

7 Serre's criterion for affineness

(Give some context, i.e. important theorem in the foundations of algebraic geometry, useful for X, Y, Z, ...)

(Discuss slight apparent mismatch to classical version regarding finiteness hypotheses)

(Discuss relative version)

The following observation is classically a fundamental result on quasi-compact and quasi-separated schemes.

Lemma 7.0.1 Let X be a scheme. Let global functions $f_1, \ldots, f_n : X \to R$ be given such that all the open subschemes $D(f_i)$ are affine and such that $X = D(f_1) \cup \ldots \cup D(f_n)$. Then the canonical map $X \to \mathrm{Spec}(R^X)$ is an open immersion. If furthermore the functions f_i generate the unit ideal in the ring R^X, this map is bijective.

Proof The classical proof, for instance as collected in the Stacks Project [**stacks**][Tag 01QF], carries over to our setting. We spell out some details.

We first show that the canonical map is injective, hence let $p, q : X$ be such that $f(p) = f(q)$ for all $f \in R^X$. Because the $D(f_i)$ cover X, one of the numbers $f_i(p)$ is invertible. Hence p and q belong to the same open $D(f_i)$. Because $D(f_i)$ is affine by assumption, we only need to verify that $g(p) = g(q)$ for all $g \in R^{D(f_i)}$ in order to conclude that $p = q$. This follows from the fact $R^{D(f_i)} = (R^X)[f_i^{-1}]$, whose classical proof (as for instance reproduced in [**liu**][Proposition 2.3.12]) just uses a bit of homological algebra and carries over to our setting verbatim.

One can check that the image of the canonical map is $D_{X'}(f_1) \cup \cdots \cup D_{X'}(f_n)$, where $D_{X'}(f_i)$ refers to the standard open of $X' := \mathrm{Spec}(R^X)$ associated to $f_i : R^X$. From this observation the remaining claims follows. $\square$

(Decide on notation for constant bundle)

Lemma 7.0.2 Let X be a scheme. Assume that $H^1(X, E) = 0$ for all wqc bundles of ideals E on X. Let $X = U \cup V$ be an open covering with U affine. Then there merely is a function $f : X \to R$ such that $D(f) \subseteq U$ and such that $X = D(f) \cup V$.

Proof As with any open subset of an affine scheme, the open subset $U \cap V$ of U is the complement of some closed subset $K \subseteq U$: If $U \cap V = D(g_1) \cup \cdots \cup D(g_m)$ for some functions $g_1, \ldots, g_m : U \to R$, we may set $K = V(g_1, \ldots, g_m)$; then $U \cap V = U \setminus K$. Let $i : K \to X$ be the inclusion map.

Let J be the subbundle of the constant bundle $\underline{R}$ with fibers $J_x = \{a : R \,|\, a \in R^\times \Rightarrow x \in U\}$. Global sections of J are global functions $f : X \to R$ such that $D(f) \subseteq U$. The bundle J is wqc by **??**.

The fibers of the pushforward bundle $i_*\underline{R}$ have the explicit description $(i_*\underline{R})_x = R^{[\![x \in K]\!]}$; a global section of this bundle is a global function $K \to R$.

We have a canonical morphism $\varphi : J \to i_*\underline{R}$, given on fibers by mapping a number $a : J_x$ to the constant map with value a. This morphism is surjective on fibers: Let $x : X$. Then $x \in U$ or $x \in V$. In the latter case, we have $x \notin K$ so $R^{[\![x \in K]\!]} = 0$. In the former case, $R^{[\![x \in K]\!]}$ is a quotient of R because the truth value of $x \in K$ is closed (if $x \in K \Leftrightarrow a_1 = \ldots = a_m = 0$, then $R^{[\![x \in K]\!]} = R/(a_1, \ldots, a_m)$) and φ_x is the canonical surjective quotient map.

Because the first cohomology of the kernel of φ vanishes, the morphism φ is also surjective on global sections. In particular, the global function $1 : K \to R$ has a preimage f. By construction of J, we have $D(f) \subseteq U$.

It remains to prove $X = D(f) \cup V$. Let $x : X$. Then $x \in U$ or $x \in V$. In the latter case, we trivially have $x \in D(f) \cup V$. In the former case, writing $x \in V \Leftrightarrow a_1 = \ldots = a_m = 0$ again, we have $\neg(f(x) = 0 \wedge a_1 = 0 \wedge \cdots \wedge a_m = 0$ by $U \cap V = U \setminus K$. By the generalized field property, one of the numbers $f(x), a_1, \ldots, a_m$ is invertible, so $x \in D(f)$ or $x \in V$. $\square$

Proposition 7.0.3 Let X be a scheme. If $H^1(X, E) = 0$ for all wqc bundles of ideals E on X, then there exist global functions as in lemma 7.0.1.

Proof Because X is a scheme, there is a finite open affine covering $X = U_1 \cup \cdots \cup U_n$. By applying lemma 7.0.2 to the binary coverings $U_1 \cup (U_2 \cup \cdots \cup U_n)$, $U_2 \cup (U_1 \cup U_3 \cup \ldots \cup U_n)$ and so on, we may assume that each open U_i is of the form $U_i = D(f_i)$ for some global function $f_i : X \to R$.

Because $X = U_1 \cup \cdots \cup U_n$, for every point $x : X$ we trivially have that the numbers $f_1(x), \ldots, f_n(x)$ generate the unit ideal in R. In other words, the bundle morphism $\underline{R}^n \to \underline{R}$ given by the matrix $(f_1 \cdots f_n)$

is surjective at each fiber. The functions $f_1, \ldots, f_n$ generate the unit ideal in R^X iff this morphism is also surjective on global sections. Hence we need to verify that its kernel has vanishing first cohomology.

If the assumption would have been that all wqc bundles of modules have vanishing first cohomology, this task would be trivial. However, the kernel K is a subbundle of $\underline{R}^n$ and hence not a bundle of ideals of $\underline{R}$. But K is filtered by its subbundles $K_j = (\{(a_1, \ldots, a_n) : K_x \mid a_i = 0 \text{ for } i > j\})_{x:X}$ for $j = 0, \ldots, n$, and each quotient K_{j+1}/K_j has vanishing first cohomology as it is isomorphic to a wqc bundle of ideals of $\underline{R}$ (by projecting to the j-th coordinate). □

Theorem 7.0.4
Let X be a scheme. If $H^1(X, E) = 0$ for all wqc bundles of ideals E on X, then X is affine.

Proof Immediate from proposition 7.0.3 and lemma 7.0.1. □

8 Application: Cohomology of Serre's twisting sheaves

Following [Har77][Theorem 5.1, Chapter III] we can apply Čech cohomology to compute the cohomology of Serre's twisting sheaves on $\mathbb{P}^n$.

Definition 8.0.1 The ring $R[X_0, \ldots, X_n]$ as well as its localizations by monomials are graded rings, where the degree $d : \mathbb{Z}$ elements are homogenous rational functions of degree d. We denote the R-module of degree d elements of $R[X_0, \ldots, X_n]_{X_{i_0} \ldots X_{i_p}}$ by

$$\left(R[X_0, \ldots, X_n]_{X_{i_0} \ldots X_{i_p}}\right)_d .$$

Theorem 8.0.2

(i) For $n : \mathbb{N}$, $d : \mathbb{Z}$, there is an isomorphism $R[X_0, \ldots, X_n]_d \to H^0(\mathbb{P}^n, \mathcal{O}(d))$ of R-modules.

(ii) $H^n(\mathbb{P}^n, \mathcal{O}(-n-1)) = R$ is free of rank 1 and $H^n(\mathbb{P}^n, \mathcal{O}(d)) = 0$ for $d > -n-1$.

(iii) The canonical map given by tensoring

$$H^0(\mathbb{P}^n, \mathcal{O}(d)) \times H^n(\mathbb{P}^n, \mathcal{O}(-d-n-1)) \to R$$

is a perfect pairing of finite free R-modules for all $d : \mathbb{Z}$.

(iv) $H^i(\mathbb{P}^n, \mathcal{O}(d)) = 0, 0 < i < n, d : \mathbb{Z}$.

Proof We cover $\mathbb{P}^n$ by the affine open subschemes $U_i :\equiv D(X_i)$. More generally, we use the shorthand $U_{i_0 \ldots i_p} :\equiv D(X_{i_0} \ldots X_{i_p})$ and note

$$\mathcal{O}(d)(U_{i_0 \ldots i_p}) = \left(R[X_0, \ldots, X_n]_{X_{i_0} \ldots X_{i_p}}\right)_d .$$

The Čech complex for this covering is

$$\prod_{i_0} \left(R[X_0, \ldots, X_n]_{X_{i_0}}\right)_d \to \prod_{i_0, i_1} \left(R[X_0, \ldots, X_n]_{X_{i_0} X_{i_1}}\right)_d \to \cdots \to \left(R[X_0, \ldots, X_n]_{X_0 \ldots X_n}\right)_d .$$

(i) $H^0(\mathbb{P}^n, \mathcal{O}(d))$ is the kernel of the first map in the Čech complex, so it consists of all families

$$C = \left(\frac{P_i}{X_i^{l_i}}\right)_{i:\{0,\ldots,n\}}$$

of degree d elements such that $X_i^{l_i} P_j = X_j^{l_j} P_i$ by regularity of X_k, $k = 0, \ldots, n$. Again by regularity and using this equation, P_i is divisible by $X_i^{l_i}$, so C was a family with values in $R[X_0, \ldots, X_n]_d$.

(ii) $H^n(\mathbb{P}^n, \mathcal{O}(d))$ is the cokernel of the map

$$\prod_i \left(R[X_0, \ldots, X_n]_{X_0 \ldots \hat{X}_i \ldots X_n}\right)_d \to \left(R[X_0, \ldots, X_n]_{X_0 \ldots X_n}\right)_d .$$

The image of this map is freely generated by all degree d monomials $X_0^{l_0} \ldots X_n^{l_n}$ where $l_i \geq 0$ for some i. This means the cokernel is generated by all degree d monomials $X_0^{l_0} \ldots X_n^{l_n}$ with $l_i < 0$ for all i. For $d = -n-1$ the only possibility is $l_i = -1$ for all i, so $H^n(\mathbb{P}^n, \mathcal{O}(-n-1))$ is freely generated by this monomial. For larger d, there is no such monomial and we have $H^n(\mathbb{P}^n, \mathcal{O}(d)) = 0$.

(iii) For $d < 0$, this is trivial by (a) and (b), so let $d \geq 0$. The pairing is given by multiplication of degree d with degree $-d-n-1$ monomials:

$$(X_0^{l_0} \dots X_n^{l_n}) \cdot (X_0^{m_0} \dots X_n^{m_n}) = X_0^{l_0+m_0} \dots X_n^{l_n+m_n}$$

where the right hand side is zero whenever there is i such that $l_i + m_i \geq 0$.

TODO: Finish, say why it is a perfect pairing and maybe what that means

(iv) TODO □

Index

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